# LINEAR STABILITY ANALYSIS OF A TWO-DIMENSIONAL ARC

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## ABSTRACT

Steady solutions to the governing equations that describe arc plasma models characterized by a two-dimensional temperature field and a one-dimensional electric field have been obtained. The stability of these steady solutions is investigated by calculating the transient created when infinitesimal changes in the electrode potentials are imposed. The results are in complete conformity with the classical Kaufmann criterion for discharge stability.

## 1. INTRODUCTION

We are concerned with the problem of the stability of two-dimensional electric arc discharges. We adopt the point of view that whether or not a circuit fault is cleared or an arc lamp survives a circuit perturbation may be determined by a consideration of the stability of the arc to small perturbations in temperature and electric field. In order to effect this stability analysis, we first obtain steady solutions to the governing equations that describe arc plasma models characterized by a two-dimensional temperature field but with a one-dimensional electric field. Such solutions, obtained through a neglect of the variations of the transverse electrostatic potential in cross sectional planes are known to approximate fully two-dimensional arc configurations to sufficient accuracy [1]. Furthermore, considerable simplification in the mathematical structure of the problem is obtained through such a description. The arc model employed accounts for variable electrical and thermal conductivities. The electrical conductivity is represented by a tanh model [2] while the variable thermal conductivity is taken into account through a Kirchhoff transformation. The temperatures and the electrostatic potentials at the electrodes are prescribed while the sidewalls enclosing the arc are taken to be electrical and heat insulators.

For arc plasma electrical conductivity modeled by a tanh function, we find that higher electrode temperatures and short arcs result in monotonically increasing current-voltage characteristics. On the other hand, lower electrode temperatures and longer arcs yield characteristics with distinct voltage maxima and minima. These characteristics look very much like those for discharge lamps shown in Waymouth [3].

There, the voltage increases steeply to a low current maximum, then decreases to a minimum as the arc expands from a thin filament to nearly fill the volume between the confining walls, and then, for still larger currents, the voltage increases nearly linearly (as for a constant resistor).

In this paper, the steady solutions are perturbed by imposing a small increment to the electrode potentials. The resulting transient is computed in time to ascertain whether the perturbed current tends to a small constant value for large time (stable) or grows asymptotically large (positive or negative) with time (unstable).

The steady and the transient solutions were obtained by finite difference formulation. The numerical results presented here are for an arc in a circuit without external resistance. The justification for this is the consideration of the arc formed during the interruption of a short circuit. For this configuration, the results agree with those of Kaufmann [4] for stability of a discharge in a circuit, that is, the negative resistance portion of the characteristic (where $dV/dI<0$) is unstable.

## 2. THEORETICAL FORMULATION

The dominant equations governing the analysis of electric arc discharges are conservation of energy and conservation of charge. Under these circumstances when the effects of convection and radiation may be neglected, the equations are: energy conservation

$$\frac{\rho c_p}{k}\frac{\partial S}{\partial t} = \nabla\cdot\nabla S + \sigma(S)\,(\nabla\phi)^2, \qquad (1)$$

and the statement that the current density $\underset{\sim}{j}$ is solenoidal

$$\nabla\cdot\underset{\sim}{j} = 0. \qquad (2)$$

In Eq. (1), S is the heat flux potential

$$S(T) = \int_0^T k(T')\,dT',$$

where k is the thermal conductivity, T is the temperature, t is the time, $\rho$ is the density, $c_p$ is the specific heat, and $\phi$ is the electrostatic potential, and $\sigma$ is the electrical conductivity. The constitutive equation for $\underset{\sim}{j}$ is Ohm's law

$$\underset{\sim}{j} = \sigma\underset{\sim}{E} \qquad (3)$$

where E is the electric field intensity. The total arc current is the cross - sectional integral of the current density,

$$I = -\int \sigma\nabla\phi\cdot d\mathcal{S}\ , \qquad (4)$$

where $d\mathcal{S}$ is an element of cross-sectional area.

The temperature variation of the transport properties and the quadratic manner in which the electric field enters the problem introduce strong nonlinearities, and any solution procedure other than a complete numerical scheme requires some simplifications. Here through S, the variation of the thermal conductivity function has been taken into account. The electrical conductivity function is represented by the following model:

$$\sigma(S) = \tfrac{1}{2}\sigma_*\{1 + \tanh[a(S-S_*)]\}, \tag{5}$$

where $\sigma_*$ and a are materials constants. The boundary conditions for the equations become

$$x=\pm L:\quad S=S_w,\ \frac{\partial\phi}{\partial x} = 0;\quad y= +\tfrac{1}{2}\ell:\ S=S_{e_2}(x),\ \phi=\phi_2;\quad y= -\tfrac{1}{2}\ell:\ S=S_{e_1}(x),\ \phi=\phi_1. \tag{6}$$

Figure 1 shows the arc geometry, boundary conditions, and the coordinate system used.

The equations are cast in dimensionless form by introducing the following variables and parameters: $\bar{S}=(S-S_*)/(S_*-S_w)$, $\mu=a(S-S_*)$, $b = \ell/2L$, $\bar{x}=x/L$, $\bar{y}=y/L$, $\bar{I}=I[2\sigma_*(S_*-S_w))]^{-1/2}$, and $\bar{\phi}=\phi[2(S_*-S_w)/\sigma_*]^{-1/2}$, where the subscript w refers to wall values. The corresponding set of equations and boundary conditions appropriate to the steady state and to a two dimensional formulation are given in [1] and will not be repeated here for brevity.

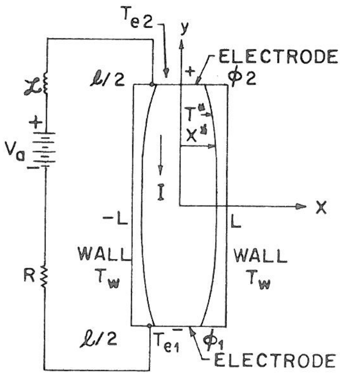


Fig. 1. Arc geometry, boundary conditions, and coordinate system.

## 3. ONE-DIMENSIONAL ELECTRIC FIELD APPROXIMATION

It is shown in [1] that solutions obtained through the neglect of the variations of the transverse electrostatic potential in cross-sectional planes are known to approximate fully two-dimensional arc configurations to sufficient accuracy. Therefore, we set $\bar{\phi} \equiv \bar{\phi}(y)$ alone. Then, omitting overbars for simplicity, for a given I, the only equation to be solved in order to determine the steady state will be:

$$\nabla^2 S + [4I^2(1 + \tanh(\mu S))/[\int_{-1}^{+1}(1 + \tanh(\mu S))dx]^2\ ] = 0, \tag{7}$$

subject to

$$x = \pm 1: S = -1; \; y = + b: S = f_{e_2}(x); \; y = -b: S = f_{e_1}(x) \tag{8}$$

The electrode temperature parameters $f_{e_2}(x)$ and $f_{e_1}(x)$ in Eq. (8) must be specified. We choose $f_{e_2}(x) = f_{20}(1 - x^2)^2 - 1$, $f_{e_1}(x)=f_{10}(1-x^2)^2-1$ (9)

where $f_{20}$ and $f_{10}$ are numerical constants so that the electrodes also satisfy the sidewall boundary conditions (additionally with zero heat conduction).

Through, $$I = - \frac{1}{2} \int_{-1}^{+1} (1+\tanh \mu S) \left(\frac{d\phi}{dy}\right) dx, \tag{10}$$

we can get, $$V_a = \int_{-b}^{+b} \left(\frac{d\phi}{dy}\right) dy , \tag{11}$$

the applied voltage. Thus the steady state is completely determined.

## 4. STABILITY THEORY

Consider a perturbation from the steady state given by

$$S = \bar{S} + S'(x,y,\tau), \; \phi = \bar{\phi} + \phi' (x,y,\tau), \; I = \bar{I} + I', \tag{12}$$

where overbar is the steady solution and prime denotes the perturbation. The corresponding energy and current conservation equations, appropriate for infinitesimally small disturbances, are:

$$\beta \frac{\partial S'}{\partial \tau} = \nabla^2 S' + \mu S' \operatorname{sech}^2 (\mu S) \left[\frac{d\phi}{dy}\right]^2 + 2[1 + \tanh(\mu S)]\left[\frac{d\phi}{dy}\right]\left[\frac{\partial \phi'}{\partial y}\right], \tag{13}$$

and

$$\nabla^2 \phi' + \mu (1 - \tanh(\mu S)) \left[\frac{\partial \phi'}{\partial x} \frac{\partial S}{\partial x} + \frac{\partial \phi'}{\partial y} \frac{\partial S}{\partial y} + \frac{d\phi}{dy} \frac{\partial S'}{\partial y}\right] - \mu^2 S' \operatorname{sech}^2(\mu S) \left[ \frac{d\phi}{dy} \frac{\partial S}{\partial y}\right] = 0, \tag{14}$$

where we have neglected squares and products of the small perturbations and have also introduced

$$\beta = \frac{\omega L^2}{\kappa}, \quad \kappa = \frac{k}{\rho c_p}, \quad \tau = \omega t.$$

$\omega$ is the dimensionless driving frequency. Additionally, we have let

$$\tanh (\mu S') \doteq (\mu S'), \tag{15}$$

and, $$[1 + \tanh (\mu(S + S'))] \doteq [1 + \tanh (\mu S)] + \mu S' \operatorname{sech}^2 (\mu S) \tag{16}$$

The perturbed current through the arc now becomes:

$$I' = -\frac{1}{2}\int_{-1}^{+1} \mu S' \operatorname{sech}^2(\mu S)\left(\frac{d\phi}{dy}\right)dx - \frac{1}{2}\int_{-1}^{+1}(1+\tanh(\mu S))\left(\frac{\partial\phi'}{\partial y}\right)dx \qquad (17)$$

The initial and boundary conditions are:

$$\tau = 0: \quad x = \pm 1: \quad S' = \frac{\partial\phi'}{\partial x} = 0; \quad y = \pm 1: \quad S' = \phi' = 0. \qquad (18)$$

$$\tau = 0^+: \quad y = \pm 1 \quad \phi' = \pm 0.1. \qquad (19)$$

The solution to both the steady state and the transient equations is effected through a finite difference procedure.

## 5. RESULTS AND DISCUSSION

Figure 2 shows the steady state $-\Delta\phi$ vs. I characteristic for b = 3. Comparison with Fig. 2 of [1] indicates that the one-dimensional approximation for the electric field is very good (for slender arcs). Numerical computations showed that S vs (y/b) at centerline also compared well with that of [1]. The transient results are shown in Figs. 3,4,5.

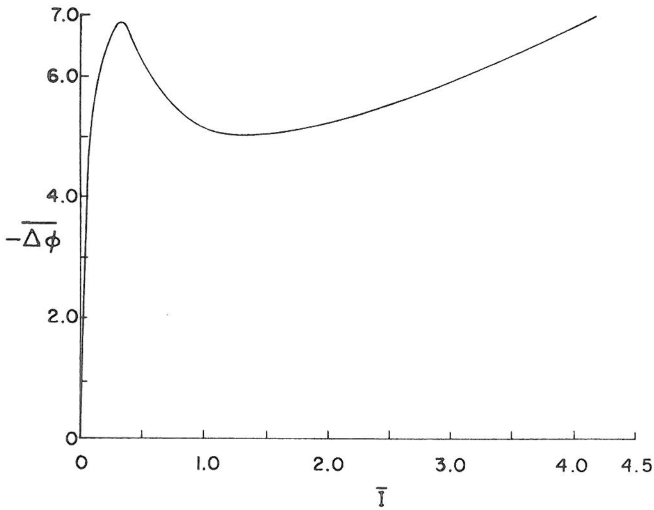


Fig. 2. Current-voltage characteristic.
$\mu = 2$, $f_{10} = 0.5$, $f_{20} = 1.1$, b = 3.

For illustration purposes, we have chosen to perturb about the following points on the steady characteristic (Fig. 2): I = 0.1, I = 0.5, and I = 4.3. These points are located on the low current increasing, the decreasing, and the high current increasing branches of the characteristic, respectively. For $\bar{I}$ = 0.1 the perturbation current corresponding to a decrease in the potential by 0.2 is shown in curve 3 of Fig. 3. The perturbation current tends asymptotically to a steady value corresponding to the appropriate decrease in current on the steady characteristic. For I = 0.5 the perturbation current corresponding to an increase in potential is shown as curve 2, and that corresponding to a decrease in potential is shown as curve 4. In the first case, the current increases rapidly to effect a transition to the high current branch while in the second case the current decreases rapidly to cause a transition to the low current branch. Operation at I=0.5 is therefore

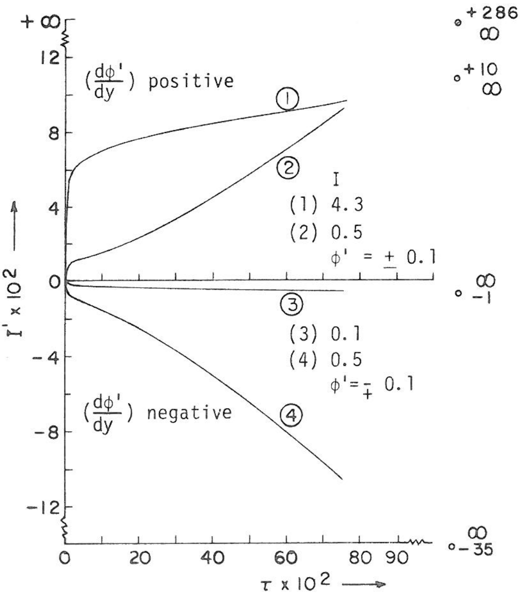


Fig. 3. I' vs. τ characteristics

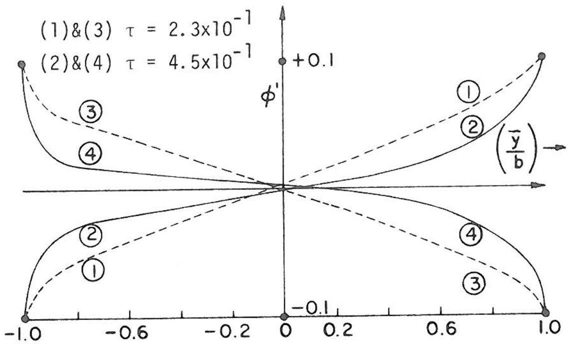


Fig. 4. φ' vs. ($\bar{y}$/b) characteristics

unstable. At I = 4.3, the perturbation current corresponding to an increase in potential is shown as curve 1 on Fig. 3. This tends asymptotically to a steady value corresponding to stable operation. These observations are in complete conformity with the classical results of Kaufmann [4]. However, heretofore, the detailed timewise behavior of arc discharges in circuits has not been calculated. Fig. 4 shows that the linear potential perturbation initially develops with time in the direction of a weak electric field in the body of the discharge with strong cathode and anode drops near the electrodes. Fig. 5 shows the transient results for the perturbation to the heat flux potential for the unstable case, I = 0.5, subject to both an increase and decrease in potential. Curves 1 and 2 show a very rapid increase in temperature through the heat flux potential which yields a rapid increase in electrical conductivity. Therefore, there is increased current leading to the transition to the high current branch. On the other hand, curves 3 and 4 show that the temperature decreases rapidly with a resultant decrease in electrical conductivity, and thus an unstable transition to the low current branch (or to extinction). We note that the increase (decrease) in heat flux potential is linear in time (for small time).

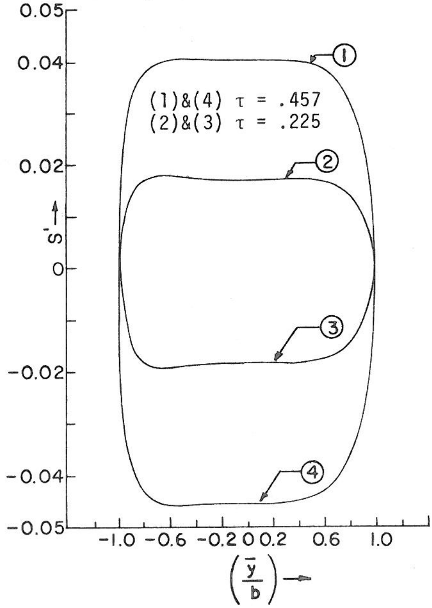


Fig. 5. S' vs. (ȳ/b) characteristics

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