STATIONARY AND NON-STATIONARY VELOCITY DISTRIBUTIONS OF HEAVY IONS IN LIGHT GASES IN ELECTRIC AND MAGNETIC FIELDS

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Abstract. A general procedure is given to obtain any time-dependent solution of the Fokker-Planck equation for heavy ions in light gases in (time-dependent) electric fields and/or static magnetic fields. In this way, both non-stationary and stationary ion velocity distributions are obtained which can be useful in the interpretation of drift-tube experiments.

The knowledge of the velocity distribution of ions in a gas in electric fields is very important for a correct interpretation of the phenomena occurring in drift tubes/1/.

Here we consider the case in which heavy ions (of mass m, charge q, and number density n), dilutely and homogeneously dispersed in an infinite gas of light atoms (of mass $M \ll m$ and number density $N \gg n$), are subject to the action of an electric field $\underline{E}(t)$ and of a static magnetic field $\underline{B}$.

We shall determine the ion velocity distribution $f(\underline{v},t)$, at time t, by solving the Boltzmann equation in the well-known approximate form of the Fokker-Planck equation

$$\frac{\partial f(\underline{v},t)}{\partial t} + \left(\underline{a}(t) + \underline{v} \times \underline{\Omega}\right) \cdot \frac{\partial f(\underline{v},t)}{\partial \underline{v}} = \lambda \left\{ div_{\underline{v}}\left(\underline{v} f(\underline{v},t)\right) + \frac{kT}{m} \nabla_{\underline{v}}^2 f(\underline{v},t) \right\} \quad (1)$$

with an assigned arbitrary initial condition. In eq.(1) $\underline{a}(t) = q\underline{E}(t)/m$ is the ion acceleration in the (time-dependent) electric field $\underline{E}(t)$, $\underline{\Omega} = q\underline{B}/m$ is the vector cyclotron frequency in the magnetic field $\underline{B}$, T is the temperature of the background gas which is assumed to be (and remain) in thermal equilibrium, k is the Boltzmann constant, and λ (which depends on the ion-neutral interaction law) is given by

$$\lambda = \frac{8}{3}\pi^2 \mu_1 \frac{M}{kT} \int_0^\infty \mathcal{F}_M(V)\, V^5 dV \int_0^\pi \sigma(V,\chi)(1-\cos\chi)\sin\chi \, d\chi . \quad (2)$$

Here, μ_1 is the mass ratio $M/(m+M)$,

$$\mathcal{F}_M(V) = N\left(\frac{M}{2\pi kT}\right)^{3/2} exp\left(-\frac{MV^2}{2kT}\right) \tag{3}$$

is the equilibrium distribution of the neutral-particle velocities V, σ is the differential scattering cross section, and χ is the scattering angle.

In order to solve eq.(1) in the most general case, we first observe that, multiplying by $\underline{v}$ both sides of eq.(1) and integrating over the whole $\underline{v}$-space, we easily obtain the equation

$$\frac{d\langle\underline{v}\rangle_t}{dt} - (\underline{a}(t) + \langle\underline{v}\rangle_t \times \underline{\Omega}) = -\lambda\langle\underline{v}\rangle_t \ , \tag{4}$$

which describes the relaxation of the mean (or drift) velocity

$$\langle\underline{v}\rangle_t = \frac{1}{n}\int_{\underline{v}} \underline{v} f(\underline{v},t)\, d\underline{v}\ . \tag{5}$$

Then we rewrite eq.(1) in the form

$$\frac{\partial f(\underline{v},t)}{\partial t} + (\underline{v} - \langle\underline{v}\rangle_t) \times \underline{\Omega} \cdot \frac{\partial f(\underline{v},t)}{\partial \underline{v}} =$$

$$= \lambda\left\{div_{\underline{v}}\left[\left(\underline{v} - \frac{1}{\lambda}[\underline{a}(t) + \langle\underline{v}\rangle_t \times \underline{\Omega}]\right) f(\underline{v},t)\right] + \frac{kT}{m}\nabla_{\underline{v}}^2 f(\underline{v},t)\right\}. \tag{6}$$

Moreover, we assume for the moment that, at any time $t \geq 0$, it is

$$(\underline{v} - \langle\underline{v}\rangle_t) \times \underline{\Omega} \cdot \frac{\partial f(\underline{v},t)}{\partial \underline{v}} = 0. \tag{7}$$

At this point we let

$$\underline{v}^* = \underline{v} - \langle\underline{v}\rangle_t\ . \tag{8}$$

Indicating with $f^*(\underline{v}^*,t)$ the ion velocity distribution in the $\underline{v}^*$-space, it is evidently

$$f(\underline{v},t) = f^*(\underline{v}^*,t) = f^*(\underline{v} - \langle\underline{v}\rangle_t, t). \tag{9}$$

Taking now into account that f^* depends on t also through $\underline{v}^*$, we have

$$\frac{\partial f(\underline{v},t)}{\partial t} = \frac{\partial f^*(\underline{v}^*,t)}{\partial t} - \frac{d\langle\underline{v}\rangle_t}{dt} \cdot \frac{\partial f^*(\underline{v}^*,t)}{\partial \underline{v}^*}\ . \tag{10}$$

Thus, observing also that

$$\frac{\partial}{\partial v_i} = \frac{\partial}{\partial v_i^*} \quad \text{and} \quad \frac{\partial^2}{\partial v_i^2} = \frac{\partial^2}{\partial v_i^{*2}} \qquad (i = x, y, z), \tag{11}$$

and making use of eq.(4), eq.(6) (subject to assumption (7)) becomes

$$\frac{\partial f^*(\underline{v}^*,t)}{\partial t} = \lambda\left\{ div_{\underline{v}^*}\left(\underline{v}^* f^*(\underline{v}^*,t)\right) + \frac{kT}{m}\nabla^2_{\underline{v}^*} f^*(\underline{v}^*,t)\right\}, \tag{12}$$

i.e. the Fokker-Planck equation for a field-free (Rayleigh) gas /2/.

The equilibrium solution $f^*_M(\underline{v}^*)$ of this equation corresponds to a solution of eq.(1) which necessarily depends on time only through $<\underline{v}>_t$. From (8) and (9) we have, in fact,

$$f(\underline{v},t) = f^*_M(\underline{v}^*) = n\left(\frac{m}{2\pi kT}\right)^{3/2} exp\left(-\frac{m(\underline{v}-<\underline{v}>_t)^2}{2kT}\right). \tag{13}$$

This result extends to the case of time-dependent electric field (but static magnetic field) the result found by Kihara/3/ for heavy ions in static electric and magnetic fields. Of course, solution (13) requires that also the initial velocity distribution $f(\underline{v},0)$ is of the same form (13) with $<\underline{v}>_0$ in place of $<\underline{v}>_t$. Moreover, the ion distribution (13) satisfies also assumption (7) at any time $t \geq 0$.

At this point, solution (13) can really be explicitly obtained once eq.(4) is solved. Assuming $\underline{\Omega}$ along the z-axis, and $\underline{a}(t) = \underline{a}_0 \cos\omega t$ in the yz-plane so that

$$\underline{a}(t) = a(t)\sin\psi\,\hat{j} + a(t)\cos\psi\,\hat{k} \qquad \left(0 \leq \psi \leq \frac{\pi}{2}\right), \tag{14}$$

we obtain

$$\begin{aligned}<v_x>_t = e^{-\lambda t}\Big\{&<v_x>_0 \cos\Omega t + <v_y>_0 \sin\Omega t\\ &-A^{-1}a_0\sin\psi\left[\Omega(\lambda^2-\omega^2+\Omega^2)\cos\Omega t + \lambda(\lambda^2+\omega^2+\Omega^2)\sin\Omega t\right]\Big\}\\ &+A^{-1}a_0\sin\psi\left[\Omega(\lambda^2-\omega^2+\Omega^2)\cos\omega t + 2\lambda\omega\Omega\sin\omega t\right],\end{aligned} \tag{15}$$

$$\begin{aligned}<v_y>_t = e^{-\lambda t}\Big\{&<v_y>_0 \cos\Omega t - <v_x>_0 \sin\Omega t\\ &-A^{-1}a_0\sin\psi\left[\lambda(\lambda^2+\omega^2+\Omega^2)\cos\Omega t - \Omega(\lambda^2-\omega^2+\Omega^2)\sin\Omega t\right]\Big\}\\ &+A^{-1}a_0\sin\psi\left[\lambda(\lambda^2+\omega^2+\Omega^2)\cos\omega t + \omega(\lambda^2+\omega^2-\Omega^2)\sin\omega t\right],\end{aligned} \tag{16}$$

$$<v_z>_t = e^{-\lambda t}\left\{<v_z>_0 - a_0\cos\psi\frac{\lambda}{\lambda^2+\omega^2}\right\} + \frac{a_0\cos\psi}{\lambda^2+\omega^2}\left[\lambda\cos\omega t + \omega\sin\omega t\right], \tag{17}$$

where $A \equiv (\lambda^2 + \omega^2 + \Omega^2)^2 - 4\omega^2\Omega^2$.

For large values of t (i.e. for $t \to \infty$, or at least for $t \gg \lambda^{-1}$) the terms involving the decreasing exponential factor vanish, and the stationary time-dependent expressions of the components of $<\underline{v}>_t$ are thus obtained. Putting moreover $\omega = 0$ (and $\underline{a}_0 \equiv \underline{a}$) the steady-state drift velocity in static electric and magnetic fields is also got. On the other hand, letting $\Omega = 0$ (or $a_0 = 0$) one can obviously eliminate the magnetic- (or electric-) field effect. Therefore, for large times, eqs.(15)-(17) are able to give a rather general expression of $<\underline{v}>_t$ to be inserted in (13) to yield the stationary ion velocity distribution, while, for any time $t > 0$, the same equations allow us to explicitly write also the non-stationary ion velocity distribution (13).

But our procedure can yield any other time-dependent solution of the Fokker-Planck eq.(1). Suppose, in fact, that (cf. eq.(9))

$$f(\underline{v},0) = f^*(\underline{v}^*,0) = n\left(\frac{m}{2\pi k T_0}\right)^{3/2} exp\left(-\frac{m\,\underline{v}^{*2}}{2kT_0}\right) \tag{18}$$

with $T_0 \neq T$. Eq.(12) can then easily be solved/2/ to give

$$f(\underline{v},t) = f^*(\underline{v}^*,t) = n\left(\frac{m}{2\pi k T_t}\right)^{3/2} exp\left(-\frac{m(\underline{v} - <\underline{v}>_t)^2}{2kT_t}\right) \tag{19}$$

with

$$T_t = T\left[1 - \left(1 - \frac{T_0}{T}\right)e^{-2\lambda t}\right], \tag{20}$$

and $<\underline{v}>_t$, as above, solution of eq.(4).

Also in this case assumption (7) is satisfied at any time, so that solution (19)-(20) is really valid. Moreover, if we take $T_0 \to 0$ in (18), we have

$$f(\underline{v},0) = n\,\delta(\underline{v} - \underline{v}_0), \tag{21}$$

where $\delta(\underline{v} - \underline{v}_0)$ is the Dirac delta function, and $\underline{v}_0 \equiv <\underline{v}>_0$. In this limit eq.(19) with

$$T_t = T\left(1 - e^{-2\lambda t}\right) \tag{22}$$

becomes the "fundamental solution" $nW(\underline{v},t;\underline{v}_0)$ of the Fokker-Planck eq.(1). Consequently, the solution (of(1)) which corresponds to any other initial ion distribution $f(\underline{v},0)$ (and which does not necessarily satisfy assumption (7)), can now be obtained by integration through the relation/2/

$$f(\underline{v},t) = \int_{\underline{v}_0} f(\underline{v}_0,0)\,W(\underline{v},t;\underline{v}_0)\,d\underline{v}_0\,. \tag{23}$$

So, the general problem of solving eq.(1) subject to an arbitrary initial condition is in principle solved, and the possibility of obtaining the heavy-ion velocity distribution in situations of practical interest in drift-tube experiments/4/ is achieved.

Of course, the limits of validity of the solution obtained here are strictly linked to those of eq.(1) itself. Thus, in the case of an initial (shifted) maxwellian distribution of the form (18) with $T_0 = T$, it is reasonable to require that (at least for $M/m \lesssim 0.02$) the condition

$$\frac{\langle \underline{v} \rangle_t^2}{kT/m} \lesssim \frac{M}{m} \tag{24}$$

must be satisfied at any time $t \geq 0$, in accordance with the conclusions of ref./5/ for the initial times, and of ref./6/ for the large times ($t \to \infty$). (Note that the sign $\sim$ must be intended here as "of the same order of magnitude as" rather than "nearly equal to").

For more details on this point, and for the solution of a kinetic equation/7/ more accurate and reliable than the Fokker-Planck equation (1), we defer the readers to future papers.

References

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